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THE DIAMETRAL DIMENSION OF THE SPACES OF WHITNEY JETS ON SEQUENCES OF POINTS

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Abstract: We calculate the diametral dimension of the spaces of Whitney jets on convergent sequences of points.

Keywords: diametral dimension, Whitney function

1. Introduction

We consider the linear topological structure of the spaces of traces of C^∞-functions on convergent sequences of points. We calculate the diametral dimension of these spaces and present a continuum of pairwise nonisomorphic spaces in the case of the so-called sparse sequences. The diametral dimension of the spaces of Whitney jets defined on thick sequences (under some condition of regularity) is the same as for the space s of rapidly decreasing sequences.

Our interest to the spaces of Whitney jets on compact sets of such a kind has arisen because of the following reasons: On the one hand, there is still no concrete example of a nuclear Fréchet function space without topological basis. The space of real analytic functions has no basis as it was proved in [1], but this space is not metrizable. The method to construct a basis for the space of Whitney functions on a convergent sequence of intervals [2] or on a sharp cusp [3] cannot be applied in our case. On the other hand, the problem of primariness is open for the spaces under examination. (The space X is primary if whenever $X = Y \bigoplus Z$ then either Y or Z is isomorphic to X.) Our spaces in a sense occupy an intermediate place between the nonprimary nuclear Fréchet spaces with continuous norm (see [4]) and the prime space $\omega = \mathbb{R}^{\mathbb{N}}$.

It should be noted that diametral dimension cannot be applied to distinguishing the spaces of Whitney jets on compact sets with nonempty interior. In fact, these spaces contain a subspace that is isomorphic to s, and so their diametral dimension is not larger than the diametral dimension of s [5, Proposition 7]. However, Mityagin showed [5] that the space s has minimal diametral dimension in the class of nuclear Fréchet spaces.

Calculation of the diametral dimension of the spaces of Whitney functions on Cantor-type sets is given in [6].

2. Preliminaries

Given a compact subset K of the real axis, let $\mathscr{E}(K)$ denote the space of all sequences $(f^{(j)}(x))_{j=0}^{\infty}$, $x \in K$, such that there exists an extension $F \in C^\infty(\mathbb{R})$ with $F^{(j)}(x) = f^{(j)}(x)$ for $j \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $x \in K$. The space $\mathscr{E}(K)$ can be identified with the quotient space $C^\infty(I)/Z$ where I is an interval containing K (let $I = [0, 1]$) and $Z = \{F \in C^\infty(I) : F^{(j)}|_K \equiv 0, j \in \mathbb{N}_0\}$. By the Whitney Theorem [7] the quotient topology can be given by the seminorms

$$\|f\|_p = |f|_p + \sup\{\big|\big(R_y^p f\big)^{(i)}(x)\big| \cdot |x-y|^{i-p} : x, y \in K,\ x \neq y,\ 0 \le i \le p\},$$

where $|f|_p = \sup\{|f^{(i)}(x)| : x \in K,\ 0 \le i \le p\}$ and $R_y^p f(x) = f(x) - \sum_{k=0}^{p} f^{(k)}(y) \frac{(x-y)^k}{k!}$ is the pth Taylor remainder, $p \in \mathbb{N}_0$.

Put $U_p = \{f \in \mathscr{E}(K) : \|f\|_p \le 1\}$.

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A Fréchet space X is said to have a continuous norm, if one of the seminorms of X is a norm. Similarly, X has no continuous norm if its every neighborhood contains a straigh line.

For each sequence $(b_k)_0^\infty$ there exists a function $F \in C^\infty(\mathbb{R})$ such that $F^{(k)}(0) = b_k$, $k \in \mathbb{N}_0$ (the Borel problem). Thus, $\mathscr{E}(\{a\})$ is isomorphic to ω for every singleton $\{a\}$.

A compact set $K \subset \mathbb{R}^m$ is C^∞-*determining* if for every C^∞-extendable function f on K with $f|_K = 0$ we have $f^{(j)}|_K = 0\ \forall j \in \mathbb{N}_0^m$.

In the one-dimensional case we trivially obtain

Proposition 1. *For a compact set K on the real axis the following are equivalent:*

(i) *K is perfect,*
(ii) *K is C^∞-determining,*
(iii) *$\mathscr{E}(K)$ has a continuous norm,*
(iv) *$\mathscr{E}(K)$ has no complemented subspace isomorphic to ω.*

We restrict exposition to the following model case of compact sets

$$K = \{0\} \cup \bigcup_{n=1}^{\infty} \{a_n\} \text{ with } a_n \searrow 0.$$

The nth Kolmogorov width of U_q with respect to U_p (see [8]) can be given as

$$d_n(U_q, U_p) = \inf_{L \in \mathscr{L}_n} \inf\{\delta : U_q \subset \delta U_p + L\}$$

where the infimum is taken over all n-dimensional subspaces of $\mathscr{E}(K)$, $n \in \mathbb{N}_0$. The diametral dimension of $X = \mathscr{E}(K)$ is defined as follows (see [9] and [5]):

$$\Gamma(X) = \{\gamma = (\gamma_n) : \forall p \exists q : \gamma_n \cdot d_n(U_q, U_p) \to 0 \text{ as } n \to \infty\}.$$

We consider the counting function corresponding to the diametral dimension

$$\beta(t) = \beta(t, U_p, U_q) := \min\{\dim L : tU_q \subset U_p + L\}, \quad t > 0.$$

It can be showed that $\beta(t) = \left|\left\{n : d_n(U_q, U_p) > \frac{1}{t}\right\}\right|$ where $|A|$ denotes the cardinality of A.

Since $\mathscr{E}(K)$ is a Schwartz space; therefore, $\beta(t, U_p, U_q)$ takes finite values for values of p and q that are sufficiently apart. The following well-known propositions express the direct relation between $\Gamma(X)$ and $\beta(t)$:

Proposition 2. $(\gamma_n) \in \Gamma(X) \iff \forall p\, \exists q : \forall C\, \exists n_0 : \beta(C\gamma_n, U_p, U_q) \le n,\ n \ge n_0.$

Proposition 3. *If Fréchet spaces X and Y are isomorphic then*

$$\forall p_1\, \exists p\, \forall q\, \exists q_1, C : \beta^{(Y)}(t, V_{p_1}, V_{q_1}) \le \beta^{(X)}(Ct, U_p, U_q), \quad t > 0,$$

and vice versa.

Here $(U_p)_{p=1}^\infty$ and $(V_p)_{p=1}^\infty$ are bases of neighborhoods of X and Y respectively.

In regard to the lower bound for Kolmogorov's widths of locally convex spaces with continuous norm, we can use the following remark of Tikhomirov (see [10] or [5, Proposition 6]).

Proposition 4. *Let U be an absolutely convex set in a linear space X and let V be an arbitrary set in X. If $\alpha U \cap L_{n+1} \subset V \cap L_{n+1}$ for some $(n+1)$-dimensional subspace L_{n+1} of X and for $\alpha > 0$ then $d_n(V, U) \ge \alpha$.*

We have to adjust this proposition for spaces without a continuous norm. In fact, if $X = \omega$ with $\|x\|_p = \max_{k \le p} |x_k|$, $p < q < r$, and $L = \mathrm{span}(e_k)_{k=r}^{r+n}$ then clearly $U_p \cap L \subset U_q \cap L$, but $d_n(U_q, U_p) = 0$ for $n \ge p$.

This example is explained by impossibility of using the Riesz Theorem (e.g., see [11, p. 84]) in this case which is essential in the proof of Proposition 4.

Let X be a locally convex space. Suppose that U is a neighborhood of zero in X. Let $Z_U = \{x \in X : \|x\|_U = 0\}$. Here $\|\cdot\|_U$ is the gauge functional of U. Let X_U be the completion of X/Z_U with respect to the norm $\|\cdot\|_U$, $\pi_U : X \to X_U : x \to \{x + Z_U\}$.

Proposition 5. *Given a set V, if $\alpha\pi_U(U) \cap M_{n+1} \subset \pi_U(V) \cap M_{n+1}$ for some $(n+1)$-dimensional subspace M_{n+1} of X_U and for $\alpha > 0$ then $d_n(V, U) \geq \alpha$.*

PROOF. Applying Proposition 4 yields $d_n^{(X_U)}(\pi(V), \pi(U)) \geq \alpha$. On the other hand, for every linear operator T we have $d_n(TV, TU) \leq d_n(V, U)$ (e.g., see [5]), which completes the proof.

Corollary 1. $\beta(t, U_p, U_q) \geq \sup\{\dim M : 2\pi_{U_p}(U_p) \cap M \subset t\pi_{U_p}(U_q)\}$ *where the supremum is taken over all finite dimensional subspaces M of X_{U_p}.*

In fact, let a subspace M with $\dim M = n + 1$ satisfy the inclusion of the hypothesis. Then $d_n(U_q, U_p) \geq \frac{2}{t}$ and, since the sequence (d_n) is nondecreasing,

$$\beta(t, U_p, U_q) \geq \left|\left\{k : d_k(U_q, U_p) \geq \frac{2}{t}\right\}\right| \geq |\{0, 1, 2, \dots, n\}| = \dim M.$$

The same argument can be repeated for every absolutely convex set U in a linear space X and for V in the linear span of U.

3. The Counting Function β in the Case of Sparse Sequences

We say that a sequence (a_n) with $a_n \searrow 0$ is *sparse* if there exists $Q \geq 1$ such that for all $n \in \mathbb{N}$

$$a_n - a_{n+1} \geq a_n^Q. \tag{1}$$

Theorem 1. *Let $K = \{0\} \cup \bigcup_{n=1}^{\infty}\{a_n\}$ be given by a sparse sequence with the corresponding constant Q. Then for the counting function corresponding to the diametral dimension of $\mathscr{E}(K)$ and for $q > p > 0$ with $q - Qp > 0$, we have*

$$N_2 \leq \beta(t, U_p, U_q) \leq (q+1)N_1, \quad t \geq 4,$$

with $N_1 = \min\{n : a_n^{q-Qp} \leq \frac{1}{8t}\}$ and $N_2 = \max\{n : (a_k - a_{k+1})^{q-p} \geq \frac{8}{t}\ \forall k \leq n\}$.

PROOF. From the definition of β we see that $\beta(t) \leq \dim L$ for every subspace L such that $tU_q \subset U_p + L$. Let us choose a suitable subspace L. We consider the following functions:

$$H_j(x) = \begin{cases} \frac{x^j}{j!} & \text{if } x \in [0, a_{N_1}] \cap K, \\ 0 & \text{otherwise on } K \end{cases}$$

and

$$h_{nj}(x) = \begin{cases} \frac{(x-a_n)^j}{j!} & \text{if } x = a_n, \\ 0 & \text{otherwise on } K \end{cases}$$

and put

$$L = \operatorname{span}\{H_j \cup h_{nj} : n = 1, \dots, N_1 - 1;\ j = 0, \dots, q\}.$$

Then $\dim L = N_1(q+1)$.

Given $f \in U_q$, we take $g \in L$ as follows:

$$g = \sum_{j=0}^{q} f^{(j)}(0)H_j + \sum_{k=1}^{N_1-1}\sum_{j=0}^{q} f^{(j)}(a_k)h_{kj}.$$

We want to show that $\|f - g\|_p \leq \frac{1}{t}$.

We see that $|f - g|_p \leq \frac{1}{2t}$. In fact, if $x > a_{N_1}$ then $f^{(i)}(x) = g^{(i)}(x)$, $i = 0, 1, \dots, p$. In case $x \leq a_{N_1}$ we have $f(x) - g(x) = R_0^q f(x)$ and $\left|R_0^q f(x)\right|_p \leq \|f\|_q x^{q-p} \leq a_{N_1}^{q-p} \leq \frac{1}{2t}$ by the choice of N_1.

In order to estimate $b_{ip} := \left|\left(R_y^p(f-g)\right)^{(i)}(x)\right| \cdot |x - y|^{i-p}$ with $x, y \in K$, $x \neq y$, $i = 0, 1, 2, \dots, p$, we will consider all possible positions of the points x, y on K.

If $x, y > a_{N_1}$ then clearly $b_{ip} = 0$.

If $x, y \le a_{N_1}$ then $(f-g)(x) = R_0^q f(x)$. Here

$$R_y^p\big(R_0^q f\big)(x) = R_y^q\big(R_0^q f\big)(x) + \sum_{k=p+1}^{q} \big(R_0^q f\big)^{(k)}(y)\frac{(x-y)^k}{k!}.$$

The first term on the right-hand side is equal to $R_y^q f(x)$, as is easy to see. Therefore,

$$\big(R_y^p(f-g)\big)^{(i)}(x) = \big(R_y^q f\big)^{(i)}(x) + \sum_{k=p+1}^{q} \big(R_0^q f\big)^{(k)}(y)\frac{(x-y)^{k-i}}{(k-i)!}$$

and

$$b_{ip} \le \|f\|_q |x-y|^{q-p} + \|f\|_q \sum_{k=p+1}^{q} \frac{y^{q-k}|x-y|^{k-p}}{(k-i)!}.$$

Since $f \in U_q$ and $y^{q-k}|x-y|^{k-p} < a_{N_1}^{q-p}$, we obtain

$$b_{ip} \le a_{N_1}^{q-p}(1+e) \le \frac{1}{2t}.$$

If $y \le a_{N_1} < x$ then $f^{(i)}(x) - g^{(i)}(x) = 0$, $f^{(k)}(y) - g^{(k)}(y) = \big(R_0^q f\big)^{(k)}(y)$ for $k = i, i+1, \ldots, p$. Therefore,

$$R_y^p(f-g)(x) = -\sum_{k=0}^{p} \big(R_0^q f\big)^{(k)}(y)\frac{(x-y)^k}{k!}$$

and

$$b_{ip} \le \|f\|_q \sum_{k=i}^{p} y^{q-k}\frac{(x-y)^{k-p}}{(k-i)!}.$$

Here $x - y \ge a_{N_1}^Q$ by (1). Hence by the definition of N_1 we have

$$b_{ip} \le \sum_{k=i}^{p} \frac{a_{N_1}^{q-k+Q(k-p)}}{(k-i)!} \le a_{N_1}^{q-Qp} e \le \frac{1}{2t}.$$

The case $x \le a_{N_1} < y$ is similar.

Therefore, $\|f-g\|_p \le \frac{1}{t}$, $U_q \subset \frac{1}{t}U_p + L$ and $\beta(t, U_p, U_q) \le (q+1)N_1$.

For the lower bound of β we use Corollary 1. In our case X_{U_p} is the Banach space $\mathscr{E}^p(K)$ of Whitney jets of order p with the norm $\|\cdot\|_p$. Put $M = \operatorname{span}\{\pi_{U_p}(h_{np}),\ n = 1, 2, \ldots, N_2\}$. We will show

$$2\pi_{U_p}(U_p) \cap M \subset t\pi_{U_p}(U_q). \tag{2}$$

Every element F on the left-hand side has the form $F = \pi_{U_p}(f)$, where all components of the jet f are zero except possibly $f^{(p)}(a_k) = \alpha_k$, $k = 1, 2, \ldots, N_2$. Since $f \in 2U_p$; therefore, $|\alpha_k| \le 2$. To prove (2) it suffices to show that $\|f\|_q \le t$. Clearly, $|f|_q \le |\alpha_k| \le \frac{t}{2}$.

Let us estimate $b_{iq} := \big|\big(R_y^q f\big)^{(i)}(x)\big| \cdot |x-y|^{i-q}$ with $x \ne y; x, y \in K$, $i \le q$. All terms of $\big(R_y^q f\big)^{(i)}(x)$ are zero, except possibly $f^{(p)}(x)$, $f^{(p)}(y)$ if $x, y \ge a_{N_2}$. If $i = p$ then $b_{pq} = |f^{(p)}(x) - f^{(p)}(y)| \cdot |x-y|^{p-q}$. If $i < p$ then $b_{iq} = |f^{(p)}(y)|\frac{|x-y|^{p-i}}{(p-i)!}|x-y|^{i-q}$.

In both cases $b_{iq} \le 2|f|_p|x-y|^{p-q} \le 4|x-y|^{p-q}$.

At least one value (x or y) is not smaller than a_{N_2} since otherwise the Taylor remainder vanishes.

Therefore, $|x-y| \ge \min_{k \le N_2}(a_k - a_{k+1}) \ge (\frac{8}{t})^{\frac{1}{q-p}}$ by the definition of N_2. This gives $b_{iq} \le \frac{t}{2}$ and (2). Thus $\beta(t) \ge \dim M = N_2$.

4. A Geometric Condition

Our next goal is to give a necessary condition for the isomorphism $\mathscr{E}(K_a) \simeq \mathscr{E}(K_b)$ in terms of the properties of the sequences (a_n) and (b_n). Let $(a_n)_{n=1}^{\infty}$ be a sparse sequence such that $a_n = \varphi(n)$ for a differentiable monotone function $\varphi : \mathbb{R}_+ \to (0,1]$. To simplify the evaluation of N_2 we suppose that φ is convex. We denote the function inverse to φ by Φ and let Φ_1 stand for the inverse to $-\varphi'$. Let the functions ψ, $\Psi = \psi_{-1}$, and $\Psi_1 = (-\psi')_{-1}$ correspond to a sparse sequence (b_n). We say that the sequences (a_n) and (b_n) are equivalent if, for each q, we can find ε, C, and x_0 such that for $x > x_0$

$$-\psi'(2qx) \le C\varphi^{\varepsilon}(x) \tag{3}$$

and the analogous condition holds on interchanging φ and ψ.

We write β_a and β_b for the counting functions corresponding to $\mathscr{E}(K_a)$ and $\mathscr{E}(K_b)$.

Given $p < q$ and large t, put $\rho = (\frac{8}{t})^{\frac{1}{q-p}}$. By the definition of N_2 we see $a_{N_2+1} - a_{N_2+2} < \rho \le a_{N_2} - a_{N_2+1}$. By the Mean Value Theorem, $-\varphi'(\xi) < \rho$ with $N_2 + 1 < \xi < N_2 + 2$. Therefore, $-\varphi'(N_2+2) < \rho$ and $N_2 + 2 > \Phi_1(\rho)$. Theorem 1 shows now that $\Phi_1(\rho) - 2 < \beta_a(t, U_p, U_q)$. In the same manner for $\rho_1 = (8t)^{-\frac{1}{q-Qp}}$ we find that

$$\beta_a(t, U_p, U_q) < (q+1)[\Phi(\rho_1) + 1]. \tag{4}$$

Applying Proposition 3, we deduce that the isomorphism $\mathscr{E}(K_a) \simeq \mathscr{E}(K_b)$ implies

$$\forall p_1\, \exists p\, \forall q\, \exists q_1\, \exists C, t_0 : \Psi_1\big((8/t)^{\frac{1}{q_1-p_1}}\big) < (q+1)\Phi((Ct)^{\frac{-1}{q-Qp}}) + q + 3, \quad t > t_0.$$

The right-hand side of the inequality can be replaced by $2q\Phi(\cdot)$ as $\Phi \uparrow \infty$ when its argument goes to 0. We now denote $\frac{1}{2q}\Psi_1((8/t)^{\frac{1}{q_1-p_1}})$ by x.

Then $8/t = (-\psi'(2qx))^{q_1-p_1}$ and $x < \Phi\big(\frac{1}{C}(-\psi'(2qx))^M\big)$ with $M = \frac{q_1-p_1}{q-Qp}$ and some constant C. This clearly implies (3).

We have thus proved the following necessary geometric condition of isomorphism:

Theorem 2. *If the spaces $\mathscr{E}(K_a)$ and $\mathscr{E}(K_b)$ are isomorphic then the sequences (a_n) and (b_n) are equivalent.*

Question. Is the equivalence of sparse sequences, provided all regularity properties, a sufficient condition for isomorphism between the corresponding spaces as well?

5. Examples of Nonisomorphic Spaces

We may now apply Proposition 2 in order to describe the diametral dimension $\Gamma(\mathscr{E}(K_a))$ for the compact set K_a satisfying all conditions of the previous section.

Proposition 6. $\big\{(\gamma_n) : \forall p\, \exists q : \gamma_n \varphi^{q-Qp}(\frac{n}{2q}) \to 0$ *as* $n \uparrow \infty\big\} \subset \Gamma(\mathscr{E}(K_a)) \subset \big\{(\gamma_n) : \forall p\, \exists q : \gamma_n(-\varphi'(n+2))^{q-p} \to 0$ *as* $n \uparrow \infty\big\}$.

Let us prove the first inclusion since the arguments are the same for the latter. If $\forall p\, \exists q : \forall C\, \exists n_0 : C\gamma_n\varphi^{q-Qp}(\frac{n}{2q}) < 1$, $n \ge n_0$, then $\frac{n}{2q} > \Phi\big((C\gamma_n)^{-\frac{1}{q-Qp}}\big)$ and $n > (q+1)\big[\Phi\big((C\gamma_n)^{-\frac{1}{q-Qp}}\big) + 1\big] > \beta\big(\frac{C\gamma_n}{8}, U_p, U_q\big)$, by (4). Therefore, by Proposition 2, $(\gamma_n) \in \Gamma(\mathscr{E}(K_a))$.

The condition (1) has the form

$$\exists Q \ge 1,\ t_0 : \varphi^Q(t) \le -\varphi'(t), \quad t > t_0.$$

If, in addition, the function φ satisfies the following restriction

$$\exists C \ge 1,\ t_1 : \varphi^C(t) \le \varphi(2t), \quad t > t_1;$$

then, as it is easy to check,

$$\Gamma(\mathscr{E}(K_a)) = \{(\gamma_n) : \exists M : \gamma_n \cdot \varphi^M(n) \to 0 \text{ as } n \uparrow \infty\}.$$

Therefore, the space of Whitney jets on the sequence (n^{-1}) has the same diametral dimension as the space s of rapidly decreasing sequences, whereas in the case $a_n = e^{-n}$ we obtain the class Γ likewise in the case of the space of entire functions.

We can now present the example of continuum-many pairwise nonisomorphic spaces $\mathscr{E}(K_{a_\lambda})$. The family of functions $\varphi_\lambda(t) = \exp(-ln^\lambda(t))$, $t \geq 1$, with the parameter $\lambda \geq 1$ (cp. [12]) gives the desired example. Indeed, the corresponding sequence is sparse, the function φ_λ satisfies all required conditions, and the classes $\Gamma(\mathscr{E}(K_{a_\lambda}))$ are different for distinct values of the parameter.

6. The Case of Thick Sequences

From the family of nonsparse sequences we distinguish the sequences (we call them *thick*) such that for every Q and large enough n we have

$$a_n - a_{n+1} \leq a_n^Q.$$

We assume the additional condition

$$\exists M, n_1 : a_n - a_{n+1} > 1/n^M \text{ for } n > n_1 \tag{5}$$

which is satisfied for typical thick sequences.

Given Fréchet spaces X and Y, we say that the functions $\beta^{(X)}$ and $\beta^{(Y)}$ have the same asymptotic behavior ($\beta^{(X)} \sim \beta^{(Y)}$) if we can estimate one function by the other with the appropriate arrangement of the quantifiers as in Proposition 3.

Theorem 3. *Let a thick convex sequence $a = (a_n)$ satisfy* (5) *and $X = \mathscr{E}(K_a)$. Then $\beta^{(X)} \sim \beta^{(s)}$.*

Proof. For the space s we have $d_n(U_q, U_p) = (n+1)^{p-q}$ (e.g., see [13, Lemma 2]) and $\beta^{(s)}(t, V_p, V_q) \sim t^{\frac{1}{q-p}}$. Since $\beta^{(s)}$ is maximal among all nuclear Fréchet spaces, we naturally obtain the upper bound for $\beta^{(X)}$:

$$\forall p\, \forall \varepsilon\, \exists q\, \exists C : \beta^{(X)}(t, U_p, U_q) < C \cdot t^\varepsilon.$$

On the other hand, arguing as in Theorem 1, we infer the bound

$$\beta^{(X)}(t, U_p, U_q) \geq N_2$$

with the same value N_2 as above. By convexity of the sequence (a_n) we have $a_{N_2+1} - a_{N_2+2} < \left(\frac{8}{t}\right)^{\frac{1}{q-p}}$. Applying (5) gives

$$a_{N_2+1} - a_{N_2+2} > (2N_2)^{-M}.$$

Therefore,

$$N_2 > Ct^{\frac{1}{M(q-p)}}$$

for some constant C, which implies the desired conclusion.

The condition (5) does not follow from the definition of a thick sequence. For example, we can recurrently construct the sequence of subscripts (n_k) such that $a_{n_k} = \frac{1}{\log n_k}$ and $a_n - a_{n+1} = n_k^{-k}$ for $n_k \leq n < n_{k+1}$. We guess that the behavior of β for the corresponding space $\mathscr{E}(K_a)$ is highly irregular.

References

1. *Domanski P. and Vogt D.*, "The space of real-analytic functions has no basis," Studia Math., **142**, No. 2, 187–200 (2000).
2. *Goncharov A. P.*, "Spaces of Whitney functions with basis," Math. Nachr., **220**, 45–57 (2000).
3. *Goncharov A. P. and Zahariuta V. P.*, "Basis in the space of C^{∞}-functions on a graduated sharp cusp," J. Geom. Anal., **13**, No. 1, 95–106 (2003).
4. *Diaz J. C.*, "On non-primary Fréchet Schwartz spaces," Studia Math., **126**, No. 3, 291–307 (1998).
5. *Mityagin B. S.*, "Approximative dimension and bases for nuclear spaces," Uspekhi Mat. Nauk, **16**, No. 4, 63–132 (1961).
6. *Arslan B., Goncharov A., and Kocatepe M.*, "Spaces of Whitney functions on Cantor-type sets," Canad. J. Math., **54**, 225–238 (2002).
7. *Whitney H.*, "Differentiable functions defined in closed sets," Trans. Amer. Math. Soc., **36**, 369–387 (1934).
8. *Kolmogorov A. N.*, "On linear dimension of topological vector spaces," Dokl. Akad. Nauk SSSR, **120**, No. 2, 239–241 (1958).
9. *Bessaga C., Pelczynski A., and Rolewicz S.*, "On diametral approximative dimension and linear homogeneity of F-spaces," Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **9**, 667–683 (1961).
10. *Tikhomirov V. M.*, "On the n-dimensional widths of compact sets," Dokl. Akad. Nauk SSSR, **130**, No. 4, 734–737 (1960).
11. *Yosida K.*, Functional Analysis, Springer-Verlag, Berlin (1978).
12. *Goncharov A. and Kocatepe M.*, "A continuum of pairwise non-isomorphic spaces of Whitney functions on Cantor-type sets," in: Linear Topological Spaces and Complex Analysis, 1997, **3**, pp. 57–64.
13. *Dragilev M. M.*, "On regular bases for nuclear spaces," Mat. Sb., **68**, No. 2, 153–173 (1965).